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FINITE CREEP DEFORMATIONS OF THICK-WALLED TUBES

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Abstract—As an application of the theory of finite deformations to creep problems, the steady-state creep of pressurized thick-walled tubes is analysed. Constitutive equations of steady-state creep in case of finite deformations are first derived by assuming the Prager–Drucker potential and Norton's law. Closed form solutions are obtained, and compared with the corresponding experiment as well as the analysis disregarding the effect of J_3 and that of infinitesimal deformations. The relation between the present analysis and that of Rimrott, performed by modifying the procedure of infinitesimal deformations, is also discussed.

1. INTRODUCTION

Since deformations of structural elements subject to creep proceed at a certain rate even under constant stress, they are generally much larger than those in elastic or elastic–plastic structures. In case of large deformations, the effect of geometrical change of the elements on the equilibrium of stress can never be neglected, and thus the usual definitions of stress, strain and their rates in infinitesimal deformation theory are no longer working definitions. Therefore, rigorous analyses of the processes of large deformations should be performed on the basis of the general theory of finite deformations. Creep problems have hardly been analysed in such a rational way, but finite deformation theory has been applied to a variety of elastic or elastic–plastic problems[3–10].

In the present paper, finite deformations in the steady-state creep¶ of pressurized thick-walled tubes are analysed as an application of the finite deformation theory to creep problems, and the process of deformations leading eventually to rupture is discussed. Though the present problem has been investigated also by Rimrott and others[1, 2], the analysis was not performed in the general way described above, but simply by modifying infinitesimal deformation analyses by taking into account the simultaneous changes of the radius and thickness of tubes. Therefore, the relations between stress, strain and their rates in Rimrott's analysis and the corresponding measures of the finite deformation theory are not always clear. Although the uniqueness and the stability bounds and the corresponding modes of deformation were elucidated by Storåkers[3] for finite deformation of visco-plastic thick-walled tubes, no information about the succeeding process leading to rupture is furnished by his investigation.

In the present paper, constitutive equations of steady-state creep in case of finite deformations are derived by assuming the Prager–Drucker potential and Norton's law in terms of

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¶ The steady-state creep here means the state wherein the constant magnitudes of stress cause the constant rates of strain.

the rate of Cauchy–Green strain tensor and the Cauchy stress tensor with respect to convected coordinates. As the creep rate is a significantly non-linear function of stresses, the effects of the third invariant of the deviatoric stress tensor J_3 in addition to the second J_2 cannot always be disregarded[23]. Thus, it would be interesting to elucidate its effect on the deformation and rupture of the tube in the case of finite deformations. The axial extension of which has been neglected in previous papers is also taken into account. The results the tube of this analysis are compared with analytical results which disregard the effect of J_3, with those obtained for infinitesimal deformations, and with the experimental results of Taira and Ohtani[32]. The relation between the present analysis and that of Rimrott is discussed.

2. FUNDAMENTAL RELATIONS

2.1 *Field equations*

Since basic equations in the theory of finite deformations have been discussed extensively in the books on continuum mechanics[8–14], only an outline necessary for the development of the succeeding analysis is given herewith.

Let us specify an element of a body by a system of convected coordinates $(\theta^1, \theta^2, \theta^3)$, and designate its position vectors in the undeformed and deformed states (at time t) by $\mathbf{r}(\theta^1, \theta^2, \theta^3)$ and $\mathbf{R}(\theta^1, \theta^2, \theta^3, t)$, respectively. Then, the base vectors $\mathbf{g}_i$, $\mathbf{G}_i$ and the metric tensors g_{ij}, G_{ij} of the coordinate system before and after deformation can be written as follows[8–11]:

$$\mathbf{g}_i = \mathbf{r}_{,i}, \qquad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \tag{1}$$

$$\mathbf{G}_i = \mathbf{R}_{,i}, \qquad G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j, \tag{2}$$

where $(\ )_{,j}$ denotes a partial derivative with respect to θ^j.

If the displacement vector $\mathbf{u}$ is expressed as

$$\mathbf{u} = \mathbf{R} - \mathbf{r} = u^i\mathbf{g}_i = U^i\mathbf{G}_i, \tag{3}$$

the Cauchy–Green strain tensor may be defined as follows[8–11]:

$$\left.\begin{aligned} \gamma_{ij} &= \tfrac{1}{2}(G_{ij} - g_{ij}) = \tfrac{1}{2}(u_i|_j + u_j|_i + u^r|_i u_r|_j) \\ &= \tfrac{1}{2}(U_i\|_j + U_j\|_i - U^r\|_i U_r\|_j), \\ \gamma^i_{.j} &= G^{ir}\gamma_{rj}, \end{aligned}\right\} \tag{4}$$

where $(\ )|_j$ and $(\ )\|_j$ represent the covariant derivatives with respect to the coordinates θ^j in the undeformed and deformed states. Since strain rates in steady-state creep are determined by the state of stress at that instant and they generally do not accompany volumetric change, the following deviatoric creep rate tensor may be conveniently adopted:

$$\left.\begin{aligned} \eta_{ij} &= \dot{\gamma}_{ij} - \tfrac{1}{3}G_{ij}G^{rs}\dot{\gamma}_{rs}, \\ \eta^i_{.j} &= G^{ik}\eta_{kj} = G^{ik}\dot{\gamma}_{kj} - \tfrac{1}{3}\delta^i_{\ j}G^{rs}\dot{\gamma}_{rs}, \end{aligned}\right\} \tag{5}$$

where $\delta^i_{\ j}$ denotes Kronecker's delta, and $(\dot{\ })$ stands for the material time derivative, which reduces to the partial derivative with respect to time in case of convected coordinates. Since the deviatoric strain rate tensor will be related to the stress tensor in the deformed state by the constitutive equations to be described later, the metric tensor of the deformed coordinates has been used to raise the indices in equations (4 and 5). For the deviatoric strain rate tensor, the following fundamental invariants may be defined:

$$I_1 = \eta^i_{.i} = 0, \qquad I_2 = \tfrac{1}{2}\eta^i_{.j}\eta^j_{.i}, \qquad I_3 = \tfrac{1}{3}\eta^i_{.j}\eta^j_{.k}\eta^k_{.i}. \tag{6}$$

When the material is incompressible, the volumetric change of a material element due to the deformation vanishes[8–11]:

$$dV - dV_0 = (\sqrt{G} - \sqrt{g})\, d\theta^1\, d\theta^2\, d\theta^3 = 0, \tag{7}$$

where $G = |G_{ij}|$ and $g = |g_{ij}|$. In this case, the relation

$$G^{rs}\dot{\gamma}_{rs} = 0 \tag{8}$$

may be verified. Thus, as observed from equation (5), the deviatoric strain rate tensor for incompressible materials coincides with the corresponding strain rate tensor also in the case of finite deformations, and we have the relations

$$\eta_{ij} = \dot{\gamma}_{ij}, \qquad \eta^{i}_{\cdot j} = G^{ik}\dot{\gamma}_{kj}. \tag{9}$$

Finally, the equations of equilibrium with respect to the deformed state may be expressed in terms of the Cauchy stress tensor (true stress) $\tau^{i}_{\cdot j}$, and these have the forms [8–14]

$$\tau^{i}_{\cdot j}\|_i + \rho(F_j - f_j) = 0, \tag{10}$$

where ρ is the density of the deformed element, and F_j and f_j denote the body force vector per unit mass and the acceleration vector at the deformed state, respectively. When the outer unit normal of the deformed surface and the traction acting on its unit area are designated by

$$\mathbf{n} = n_i \mathbf{G}^i, \qquad \mathbf{P} = P_i \mathbf{G}^i = P^i \mathbf{G}_i,$$

the boundary conditions for equation (10) are given as follows[8–14]:

$$\tau^{ij} n_i = P^j \quad \text{or} \quad \tau^{i}_{\cdot j} n_i = P_j. \tag{11}$$

Since the creep of metals is generally insensitive to hydrostatic pressure, it is convenient to introduce deviatoric stress tensors π^{ij} or $\pi^{i}_{\cdot j}$, which are related to τ^{ij} as follows:

$$\pi^{ij} = \tau^{ij} - \tfrac{1}{3}G^{ij}\tau^{r}_{\cdot r}, \qquad \pi^{i}_{\cdot j} = \tau^{i}_{\cdot j} - \tfrac{1}{3}\delta^{i}_{j}\tau^{r}_{\cdot r}. \tag{12}$$

For $\pi^{i}_{\cdot j}$ in the above relation, the following fundamental invariants may be defined:

$$J_1 = \pi^{i}_{\cdot i} = 0, \qquad J_2 = \tfrac{1}{2}\pi^{i}_{\cdot j}\pi^{j}_{\cdot i}, \qquad J_3 = \tfrac{1}{3}\pi^{i}_{\cdot j}\pi^{j}_{\cdot k}\pi^{k}_{\cdot i}. \tag{13}$$

2.2 *Constitutive equations*

In contrast to the infinitesimal deformation, the stress–strain or stress–strain rate relation in the finite deformation theory is subject to various additional restrictions. Hence, for elastic-plastic and elastic–viscoplastic materials, general forms of constitutive equations have been discussed from the mathematic and thermodynamic points of view[8–21], and are specialized to various problems[3–10,17,21]. Regarding the finite deformations of non-linear creep, however, we still lack derivations of constitutive equations of practical forms obtained by specializing the general equations in conformity with experimental observations.

In the following, therefore, a constitutive equation of steady-state finite creep deformations for isotropic incompressible materials will be derived by assuming the Norton creep law and the Prager–Drucker potential which includes both the effects of J_3 and J_2. The constitutive relation in finite deformations should remain unchanged if the body is subjected to an arbitrary rigid-body motion[8–14]. This requirement is satisfied automatically if we use the stress, strain and their rates with respect to the convected coordinates, and so these are employed in the following derivation.

The effect of hydrostatic pressure on creep of metals is generally small, and can be neglected with sufficient accuracy[22, 27]. Furthermore, since steady-state creep is not accompanied by appreciable volumetric strain, the deviatoric strain rate tensor η_{ij} for the isotropic material may be expressed as an isotropic tensor function of the deviatoric stress tensor π^{ij}. Then, if creep potential f is assumed to exist for η_{ij}, we have the relation

$$\eta_{ij} = \mu \frac{\partial f(J_2, J_3)}{\partial \tau^{ij}}, \tag{14}$$

where μ denotes a certain scalar coefficient, and f is a scalar function of J_2 and J_3 only. It is difficult to obtain rigorous proof for existence of the creep potential in equation (14). However, it has been proved at least as far as polycrystalline slip models for infinitesimal deformation are concerned[24, 25], and it has also been confirmed experimentally[26]. Since it is postulated tacitly that the body is subject to load from natural state and the variation of stress during the creep process is sufficiently small, the creep rate practically depends only on the state of stress at the given instant. Hence, the assumption of a creep potential seems valid enough for the present problem.

In order to bring equation (14) into conformity with the experimental results, it is necessary to define an equivalent stress σ_e and an equivalent strain rate η_e, besides specifying the forms of creep potential f. For isotropic incompressible materials, these equivalent quantities may be expressed as functions of the second and third invariants of $\pi^{i}_{\cdot j}$ and $\eta^{i}_{\cdot j}$:

$$\sigma_e = P(J_2, J_3), \qquad \eta_e = Q(I_2, I_3). \tag{15}$$

Symbols P and Q in the above relations are homogeneous functions with the same dimensions as the corresponding deviatoric tensors. Quantities σ_e and η_e in equation (15) should specify the equivalence of stress and creep rate in multiaxial states in relation to those in uniaxial states[23]. However, if the coefficient μ in equation (14) is chosen as

$$\mu = \frac{\bar{\mu}(\sigma_e)}{M}, \qquad M = Q\left(\frac{1}{2}\frac{\partial f}{\partial \tau^{i}_{\cdot j}}\frac{\partial f}{\partial \tau^{j}_{\cdot i}}, \frac{1}{3}\frac{\partial f}{\partial \tau^{i}_{\cdot j}}\frac{\partial f}{\partial \tau^{j}_{\cdot k}}\frac{\partial f}{\partial \tau^{k}_{\cdot i}}\right), \tag{16}$$

then equations (14) and (15) yield the relation

$$\eta_e = \bar{\mu}(\sigma_e). \tag{17}$$

Hence the above mentioned condition is always satisfied[23, 29].

Now, let us assume that the results of creep tests in uniaxial tension can be described with sufficient accuracy by Norton's law[27] expressed in terms of the Cauchy stress tensor and the strain rate tensor for convected coordinates†

† This assumption may be rephrased as Norton's law relating logarithmic strain rate $\dot{\varepsilon}_x$ and true stress σ_x in uniaxial tension. In fact, according to equations (9 and 34), we have the relation

$$\dot{\varepsilon}_x = \frac{\partial}{\partial t}\ln\left(1 + \frac{\partial u_x}{\partial x}\right) = \frac{\dfrac{\partial \dot{u}_x}{\partial x}}{1 + \dfrac{\partial u_x}{\partial x}} = \eta^{1}_{\cdot 1}, \qquad \sigma_x = \tau^{1}_{\cdot 1}$$

and hence equation (18) may be written in the alternative form

$$\dot{\varepsilon}_x = k\sigma_x^{\,n}.$$

$$\eta^{1}_{\cdot 1} = k(\tau^{1}_{\cdot 1})^n, \tag{18}$$

where k and n are creep constants. Equation (18) has been ascertained to be valid if the relevant levels of stress are not too high[27, 28]. As mentioned above, the equivalent creep rate and the equivalent stress are related by the same equation as the uniaxial states. Equation (18), therefore, holds also for η_e and σ_e:

$$\eta_e = k\sigma_e^{\,n}. \tag{19}$$

Then, let us assume the Prager–Drucker potential[30, 31]

$$f = (J_2)^3 - c(J_3)^2 \tag{20}$$

as the most fundamental one which includes both J_3 and J_2. The symbol c in the above equation stands for a material constant. The potential of equation (20) may be interpreted as that derived by expanding a function of J_2 and J_3 of even order regarding stress (but otherwise of arbitrary form), by assuming the effect of J_3 to be smaller than that of J_2. Accordingly, if the material constant c is determined by the pure torsion or the internal pressure test of tube specimens where the effect of J_3 is most significant, the potential (20) may have enough generality[23]. Then, equation (14), together with equations (16) and (20), furnish

$$\left.\begin{aligned} \eta_{ij} &= \frac{\bar{\mu}}{M} G_{ir}[3(J_2)^2\pi^{rs} - 2cJ_3 t^{rs}]G_{sj}, \\ t^{ij} &= \pi^{ir}G_{rs}\pi^{sj} - \tfrac{2}{3}G^{ij}J_2, \end{aligned}\right\} \tag{21}$$

where t^{ij} stands for a deviator of the square of the deviatoric stress tensor.

If we assume that the equivalent stress function P in equation (15) is identical with the potential f up to a certain power and a certain numerical coefficient, the equivalent stress may be defined as

$$\sigma_e = \sqrt{3}\left[\frac{(J_2)^3 - c(J_3)^2}{1 - \frac{4}{27}c}\right]^{1/6}. \tag{22}$$

When the equivalent creep rate η_e may be expressed as a function of the second invariant $\eta^{i}_{\cdot j}$ only, we have the relation

$$\eta_e = \frac{2}{\sqrt{3}}(I_2)^{1/2}. \tag{23}$$

In view of equations (15) and (16), M in equation (21) may be furnished from equation (20)

$$M = \frac{2}{\sqrt{3}}\left(\frac{1}{2}\frac{\partial f}{\partial \tau^{i}_{\cdot j}}\frac{\partial f}{\partial \tau^{j}_{\cdot i}}\right)^{1/2} = 2\sqrt{3}[(J_2)^3 - (2 - \tfrac{4}{27}c)c(J_3)^2]^{1/2}J_2. \tag{24}$$

Then, the relation (17) may be verified actually, and comparison of equation (17) with (19) gives

$$\bar{\mu} = k\sigma_e^{\,n}. \tag{25}$$

By substituting this into equation (21), we finally obtain the constitutive equation of finite creep deformations incorporating the effect of J_3:

$$\eta_{ij} = \frac{\sqrt{3}}{2} k\sigma_e{}^n G_{ir} \left\{ \frac{(J_2)^2 \pi^{rs} - \frac{3}{2} c J_3 t^{rs}}{[(J_2)^3 - (2 - \frac{4}{27}c)c(J_3)^2]^{1/2} J_2} \right\} G_{sj}. \tag{26}$$

In case of $c = 0$, in particular, the above equation has the form

$$\eta_{ij} = \tfrac{3}{2} k\sigma_e{}^{n-1} G_{ir} \pi^{rs} G_{sj}, \qquad \sigma_e = (3J_2)^{1/2}, \tag{27}$$

which is an extension of the Mises type equation to finite deformations.

3. FINITE CREEP ANALYSIS OF PRESSURIZED THICK-WALLED TUBES

Consider a thick-walled tube with closed ends under internal and external pressure p_i and p_0, and let the inner and outer radii in the undeformed state be a and b, respectively. Let us adopt a cylindrical coordinate system (r, θ, z) as the convected coordinates.

In analyses of infinitesimal deformations of pressurized tubes, a state of plane stain in the axial direction has been assumed. However, we have no previous certitude as to the validity of the assumption in the case of finite deformations. Since the tube is sufficiently long, the state of generalized plane strain[10] prevails over the entire tube except in the vicinity of the closed ends. Because of the axisymmetry of the problem, the circumferential displacement component always vanishes. Hence, the axial and radial components of displacement u_r, u_z may be written as

$$u_r = u_r(r, t), \qquad u_z = [\lambda(t) - 1]z, \tag{28}$$

where $\lambda(t)$ denotes the extension in the axial direction.

The metric tensors of the convected coordinate system of the undeformed and deformed body may now be given as follows:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{29a}$$

$$G_{ij} = \begin{bmatrix} \left(1 + \dfrac{\partial u_r}{\partial r}\right)^2 & 0 & 0 \\ 0 & (r + u_r)^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}, \quad G^{ij} = \begin{bmatrix} \left(1 + \dfrac{\partial u_r}{\partial r}\right)^{-2} & 0 & 0 \\ 0 & (r + u_r)^{-2} & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}. \tag{29b}$$

According to equation (29b), the non-zero components of the Christoffel symbol of the second kind for the convected coordinates of the deformed body may be expressed as

$$\left.\begin{aligned} \Gamma^1_{11} &= \frac{\partial^2 u_r}{\partial r^2} \bigg/ \left(1 + \frac{\partial u_r}{\partial r}\right), \qquad \Gamma^1_{22} = -(r + u_r) \bigg/ \left(1 + \frac{\partial u_r}{\partial r}\right), \\ \Gamma^2_{12} &= \Gamma^2_{21} = \left(1 + \frac{\partial u_r}{\partial r}\right) \bigg/ (r + u_r). \end{aligned}\right\} \tag{30}$$

By means of equations (29), the condition (7) leads to

$$(r + u_r)^2 = \frac{r^2 + a^2 H(t)}{\lambda(t)}. \tag{31}$$

$H(t)$ in the above relations denotes an arbitrary function of time t. The non-zero components of the strain and strain rate tensor calculated from equations (4, 9, 29 and 31) are

$$\gamma^{1}_{\cdot 1} = \frac{1}{2}\left\{1 - \lambda\left[1 + \left(\frac{r}{a}\right)^{-2} H\right]\right\}, \quad \gamma^{2}_{\cdot 2} = \frac{1}{2}\left[1 - \frac{\lambda}{1 + \left(\frac{r}{a}\right)^{-2} H}\right], \quad \gamma^{3}_{\cdot 3} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right), \tag{32}$$

$$\eta^{1}_{\cdot 1} = -\frac{1}{2}\left[\frac{\dot{\lambda}}{\lambda} + \frac{\dot{H}}{\left(\frac{r}{a}\right)^2 + H}\right], \quad \eta^{2}_{\cdot 2} = -\frac{1}{2}\left[\frac{\dot{\lambda}}{\lambda} - \frac{\dot{H}}{\left(\frac{r}{a}\right)^2 + H}\right], \quad \eta^{3}_{\cdot 3} = \frac{\dot{\lambda}}{\lambda}. \tag{33}$$

The physical components of the Cauchy stress tensor $\tau^{i}_{\cdot j}$ different from zero are $\sigma^{r}_{\cdot r}$, $\sigma^{\theta}_{\cdot \theta}$ and $\sigma^{z}_{\cdot z}$, which in this case are given by

$$\left.\begin{aligned} \sigma^{r}_{\cdot r} &= \tau^{11}G_{11} = \tau^{1}_{\cdot 1}, \\ \sigma^{\theta}_{\cdot \theta} &= \tau^{22}G_{22} = \tau^{2}_{\cdot 2}, \\ \sigma^{z}_{\cdot z} &= \tau^{33}G_{33} = \tau^{3}_{\cdot 3}. \end{aligned}\right\} \tag{34}$$

The equation of equilibrium for the deformed state (10) reduces to a single relation, which becomes, by means of equation (31),

$$\frac{\partial \tau^{1}_{\cdot 1}}{\partial r} = \frac{r}{a^2}\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1}(\tau^{2}_{\cdot 2} - \tau^{1}_{\cdot 1}). \tag{35}$$

The equilibrium condition of axial forces, on the other hand, may be expressed in the spatial description as

$$\pi[a + (u_r)_a]^2 p_i - \pi[b + (u_r)_b]^2 p_0 = \int_{a+(u_r)a}^{b+(u_r)b} \sigma^{z}_{\cdot z}(2\pi)(r + u_r)\,\mathrm{d}(r + u_r).$$

By means of equation (31), this is transformed into an alternative form of the material description:

$$(1 + H)a^2 p_i - \left[\left(\frac{b}{a}\right)^2 + H\right]a^2 p_0 = 2\int_a^b \tau^{3}_{\cdot 3} r\,\mathrm{d}r. \tag{36}$$

The constitutive equation (26) in the present problem may be written as

$$-\frac{1}{2}\left[\frac{\dot{\lambda}}{\lambda} + \frac{\dot{H}}{\left(\frac{r}{a}\right)^2 + H}\right] = \frac{\bar{\mu}}{M}\,[3(J_2)^2\pi^{1}_{\cdot 1} - 2cJ_3\,t^{1}_{\cdot 1}], \tag{37a}$$

$$-\frac{1}{2}\left[\frac{\dot{\lambda}}{\lambda} - \frac{\dot{H}}{\left(\frac{r}{a}\right)^2 + H}\right] = \frac{\bar{\mu}}{M}\,[3(J_2)^2\pi^{2}_{\cdot 2} - 2cJ_3\,t^{2}_{\cdot 2}], \tag{37b}$$

$$\frac{\dot{\lambda}}{\lambda} = \frac{\bar{\mu}}{M}\,[3(J_2)^2\pi^{3}_{\cdot 3} - 2cJ_3\,t^{3}_{\cdot 3}] = \frac{\bar{\mu}}{M}\,[3(J_2)^2 - 2c\pi^{1}_{\cdot 1}\pi^{2}_{\cdot 2}\,t^{3}_{\cdot 3}]\pi^{3}_{\cdot 3}, \tag{37c}$$

where M and $\bar{\mu}$ are given by equations (24) and (25).

By use of equation (12), equation (37c) becomes

$$\tau^3_{\cdot 3} = \tau^1_{\cdot 1} + \tfrac{1}{2}(\tau^2_{\cdot 2} - \tau^1_{\cdot 1}) + \frac{3\dot{\lambda}M}{2\lambda\bar{\mu}[3(J_2)^2 - 2c\pi^1_{\cdot 1}\pi^2_{\cdot 2}t^3_{\cdot 3}]}.$$

By eliminating the term $(\tau^2_{\cdot 2} - \tau^1_{\cdot 1})$ of the above relation from equation (35) and substituting the resulting relation into equation (36), we have

$$(1 + H)a^2 p_i - \left[\left(\frac{b}{a}\right)^2 + H\right] a^2 p_0 = \int_a^b \frac{\partial}{\partial r}[(r^2 + a^2 H)\tau^1_{\cdot 1}]\,dr + 3\frac{\dot{\lambda}}{\lambda}\int_a^b \frac{Mr}{\bar{\mu}[3(J_2)^2 - 2c\pi^1_{\cdot 1}\pi^2_{\cdot 2}t^3_{\cdot 3}]}\,dr. \tag{38}$$

Application of the boundary conditions, i.e.

$$\left.\begin{aligned} \sigma^r_{\cdot r} &= \tau^1_{\cdot 1} = -p_i, &\quad \text{at } r = a, \\ \sigma^r_{\cdot r} &= \tau^2_{\cdot 2} = -p_0, &\quad \text{at } r = b \end{aligned}\right\} \tag{39}$$

to the first integral of the right-hand side of equation (38) gives

$$\frac{\dot{\lambda}}{\lambda}\int_a^b \frac{Mr}{\bar{\mu}[3(J_2)^2 - 2c\pi^1_{\cdot 1}\pi^2_{\cdot 2}t^3_{\cdot 3}]}\,dr = 0. \tag{40}$$

Since the integral in this relation does not vanish for arbitrary values of stress, we obtain the following relations

$$\dot{\lambda}(t) = 0, \qquad \lambda(t) = 1. \tag{41}$$

Thus, the assumption of plane strain in the axial direction is proved valid for the steady-state creep of pressurized closed tubes also in the case of finite deformations.†

According to equation (41), equations (37c) and (13) give the relations

$$\pi^3_{\cdot 3} = 0, \qquad J_3 = 0. \tag{42}$$

Then, equations (22) and (37) have the forms

$$\sigma_e = \frac{\sqrt{3}}{2}(1 - \tfrac{4}{27}c)^{-1/6}|\tau^2_{\cdot 2} - \tau^1_{\cdot 1}|, \tag{43}$$

$$\frac{\dot{H}}{2}\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1} = \operatorname{sign}(\tau^2_{\cdot 2} - \tau^1_{\cdot 1})\cdot\left(\frac{\sqrt{3}}{2}\right)^{n+1} k(1 - \tfrac{4}{27}c)^{-n/6}|\tau^2_{\cdot 2} - \tau^1_{\cdot 1}|^n. \tag{44}$$

By eliminating the term $(\tau^2_{\cdot 2} - \tau^1_{\cdot 1})$ in equation (35) from equation (44) and integrating it with respect to time, we have the relation

$$\tau^1_{\cdot 1} = -\operatorname{sign}(\tau^2_{\cdot 2} - \tau^1_{\cdot 1})\cdot\frac{n}{\sqrt{3}}(1 - \tfrac{4}{27}c)^{1/6}\left[\frac{\operatorname{sign}(\tau^2_{\cdot 2} - \tau^1_{\cdot 1})\cdot\dot{H}}{\sqrt{3}k}\right]^{1/n}\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1/n} + F(t), \tag{45}$$

where $F(t)$ denotes an arbitrary function of time t. According to the boundary condition (39), $\dot{H}(t)$ and $F(t)$ in the above equations are determined as

† In view of equations (37c)–(41), it may be easily proved that the state of plane strain generally occurs in the pressurized closed tubes if their creep behaviour is governed by constitutive equations of the form

$$\eta_{ij} = G_{ir}(A\pi^{rs} + BJ_3 t^{rs})G_{sj},$$

where A and B are arbitrary scalar functions of stress and strain. Equations derived from the creep potential $f(J_2, (J_3)^2)$ are examples of the equations of such a form.

$$\dot{H}(t) = \operatorname{sign}(\tau^2_{\cdot 2} - \tau^1_{\cdot 1}) \cdot \sqrt{3}k \left[\frac{\operatorname{sign}(\tau^2_{\cdot 2} - \tau^1_{\cdot 1}) \cdot \frac{\sqrt{3}}{n}(p_i - p_0)}{(1 - \frac{4}{27}c)^{1/6}\left\{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}\right\}} \right]^n, \tag{46a}$$

$$F(t) = \frac{-p_0(1+H)^{-1/n} + p_i\left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}. \tag{46b}$$

Substitution of this into equations (42, 44 and 45) furnishes the components of stress:

$$\left.\begin{aligned}
\tau^1_{\cdot 1} &= \frac{-p_0(1+H)^{-1/n} - (p_i - p_0)\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1/n} + p_i\left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}},\\
\tau^2_{\cdot 2} &= \frac{-p_0(1+H)^{-1/n} - \left(1 - \frac{2}{n}\right)(p_i - p_0)\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1/n} + p_i\left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}},\\
\tau^3_{\cdot 3} &= \frac{-p_0(1+H)^{-1/n} - \left(1 - \frac{1}{n}\right)(p_1 - p_0)\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1/n} + p_i\left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}.
\end{aligned}\right\} \tag{47}$$

From the above equations, we have

$$\operatorname{sign}(\tau^2_{\cdot 2} - \tau^1_{\cdot 1}) = \operatorname{sign}(p_i - p_0). \tag{48}$$

The expressions of displacement, strain and strain rate, furthermore, are obtained from equations (31–33, 41, 46 and 48):

$$\left.\begin{aligned}
&u_r = (r^2 + a^2H)^{1/2} - r,\\
&\gamma^1_{\cdot 1} = -\frac{1}{2}\left(\frac{r}{a}\right)^{-2} H, \qquad \gamma^2_{\cdot 2} = \frac{1}{2}\left\{1 - \left[1 + \left(\frac{r}{a}\right)^{-2} H\right]^{-1}\right\}, \qquad \gamma^3_{\cdot 3} = 0,\\
&\eta^2_{\cdot 2} = -\eta^1_{\cdot 1} = \operatorname{sign}(p_i - p_0) \cdot \frac{\sqrt{3}}{2}k\\
&\qquad \times \left[\frac{\frac{\sqrt{3}}{n}|p_i - p_0|\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1/n}}{(1 - \frac{4}{27}c)^{1/6}\left\{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}\right\}} \right]^n, \qquad \eta^3_{\cdot 3} = 0.
\end{aligned}\right\} \tag{49}$$

The variable $H(t)$ in the preceding relations may be determined from equation (46a). Since equation (49) gives the relation $H(0) = 0$, through use of equation (48) equation (46a) becomes

$$\int_0^H \left\{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}\right\}^n \mathrm{d}H$$
$$= \sqrt{3}k\left(\frac{\sqrt{3}}{n}\right)^n (1 - \tfrac{4}{27}c)^{-n/6} \int_0^t \mathrm{sign}(p_i - p_0)\cdot |p_i - p_0|^n\,\mathrm{d}t. \quad (50)$$

By introducing the transformation

$$H = \frac{\left(\frac{b}{a}\right)^2 x^n - 1}{1 - x^n}, \quad (51)$$

this reduces to a more concise form:

$$n\int_{\left(\frac{b}{a}\right)^{-2/n}}^{x} \frac{(1-x)^n}{x(1-x^n)}\,\mathrm{d}x = \sqrt{3}k\left(\frac{\sqrt{3}}{n}\right)^n (1 - \tfrac{4}{27}c)^{-n/6} \int_0^t \mathrm{sign}(p_i - p_0)\cdot |p_i - p_0|^n\,\mathrm{d}t. \quad (52)$$

For positive integer n, in particular, the left-hand side can be integrated analytically[2].

Once the value of H has been determined as a function of t from equations (51) and (52), the states of stress and strain in the tubes can be determined by equations (47) and (49) as functions of t.

When pressures p_i and p_0 are constant, specifically, we can introduce nondimensional quantities

$$\left.\begin{aligned} &T^i_{\cdot j} = \frac{\tau^i_{\cdot j}}{p_i}, \quad U_r = \frac{u_r}{a}, \quad T = \mathrm{sign}(p_i - p_0)\cdot\frac{\sqrt{3}}{2}k\left\{\frac{\frac{\sqrt{3}}{n}|p_i - p_0|}{(1-\frac{4}{27}c)^{1/6}\left[1 - \left(\frac{b}{a}\right)^{-2/n}\right]}\right\}^n t,\\ &\Gamma^i_{\cdot j} = \gamma^i_{\cdot j}, \quad H^i_{\cdot j} = \frac{\eta^i_{\cdot j}}{\mathrm{sign}\,(p_i - p_0)\cdot\frac{\sqrt{3}}{2}k\left\{\frac{\frac{\sqrt{3}}{n}|p_i - p_0|}{(1-\frac{4}{27}c)^{1/6}\left[1 - \left(\frac{b}{a}\right)^{-2/n}\right]}\right\}^n}, \end{aligned}\right\} \quad (53)$$

and equations (47)–(50) reduce to simpler forms

$$\left.\begin{aligned} T^1_{\cdot 1} &= \frac{-\frac{p_0}{p_i}(1+H)^{-1/n} - \left(1 - \frac{p_0}{p_i}\right)\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1/n} + \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}},\\ T^2_{\cdot 2} &= \frac{-\frac{p_0}{p_i}(1+H)^{1/n} - \left(1 - \frac{2}{n}\right)\left(1 - \frac{p_0}{p_i}\right)\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1/n} + \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}},\\ T^3_{\cdot 3} &= \frac{-\frac{p_0}{p_i}(1+H)^{-1/n} - \left(1 - \frac{1}{n}\right)\left(1 - \frac{p_0}{p_i}\right)\left[\left(\frac{r}{a}\right)^2 + H\right]^{-1/n} + \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}, \end{aligned}\right\} \quad (54)$$

$$U_r = \left[\left(\frac{r}{a}\right)^2 + H\right]^{1/2} - \frac{r}{a}, \tag{55}$$

$$\Gamma^1_{\cdot 1} = -\frac{1}{2}\left(\frac{r}{a}\right)^{-2} H, \qquad \Gamma^2_{\cdot 2} = \frac{1}{2}\left\{1 - \left[1 + \left(\frac{r}{a}\right)^{-2} H\right]^{-1}\right\}, \qquad \Gamma^3_{\cdot 3} = 0, \tag{56}$$

$$H^2_{\cdot 2} = -H^1_{\cdot 1} = \frac{1}{\left(\frac{r}{a}\right)^2 + H}\left\{\frac{1 - \left(\frac{b}{a}\right)^{-2/n}}{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}\right\}^n, \qquad H^3_{\cdot 3} = 0, \tag{57}$$

$$\frac{1}{2}\int_0^H \left\{\frac{(1+H)^{-1/n} - \left[\left(\frac{b}{a}\right)^2 + H\right]^{-1/n}}{1 - \left(\frac{b}{a}\right)^{-2/n}}\right\}^n \mathrm{d}H = T. \tag{58}$$

Accordingly, it will be observed that the results of the analysis may be completely specified by the three parameters n, b/a and p_0/p_i.

In the limiting case of $H \to 0$ and $H \to 2T$, the preceding relations reduce to the analytical results of infinitesimal deformations

$$\left.\begin{aligned}
T^1_{\cdot 1} &= \frac{-\frac{p_0}{p_i}a^{-2/n} - \left(1 - \frac{p_0}{p_i}\right)r^{-2/n} + b^{-2/n}}{a^{-2/n} - b^{-2/n}},\\
T^2_{\cdot 2} &= \frac{-\frac{p_0}{p_i}a^{-2/n} - \left(1 - \frac{2}{n}\right)\left(1 - \frac{p_0}{p_i}\right)r^{-2/n} + b^{-2/n}}{a^{-2/n} - b^{-2/n}},\\
T^3_{\cdot 3} &= \frac{-\frac{p_0}{p_i}a^{-2/n} - \left(1 - \frac{1}{n}\right)\left(1 - \frac{p_0}{p_i}\right)r^{-2/n} + b^{-2/n}}{a^{-2/n} - b^{-2/n}},
\end{aligned}\right\} \tag{59}$$

$$U_r = \mathrm{sign}(p_i - p_0)\frac{\sqrt{3}}{2}k\left(\frac{r}{a}\right)^{-1}\left\{\frac{\frac{\sqrt{3}}{n}|p_i - p_0|}{(1 - \frac{4}{27}c)^{1/6}\left[1 - \left(\frac{b}{a}\right)^{-2/n}\right]}\right\}^n t, \tag{60}$$

$$\Gamma^2_{\cdot 2} = -\Gamma^1_{\cdot 1} = \mathrm{sign}(p_i - p_0)\cdot\frac{\sqrt{3}}{2}k\left(\frac{r}{a}\right)^{-2}\left\{\frac{\frac{\sqrt{3}}{n}|p_i - p_0|}{(1 - \frac{4}{27}c)^{1/6}\left[1 - \left(\frac{b}{a}\right)^{-2/n}\right]}\right\}^n t, \quad \Gamma^3_{\cdot 3} = 0, \tag{61}$$

$$H^2_{\cdot 2} = -H^1_{\cdot 1} = \left(\frac{r}{a}\right)^{-2}, \qquad H^3_{\cdot 3} = 0. \tag{62}$$

If we take $c = 0$ in nondimensional quantities (53), equations (59)–(62) lead to the well-known results according to the Mises type theory[27].

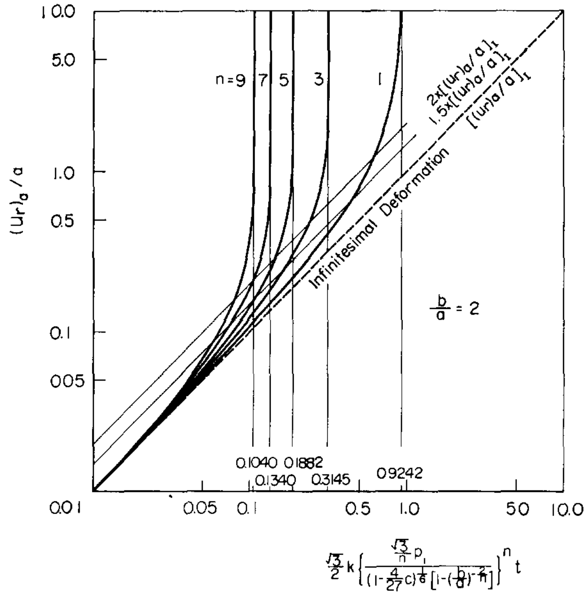
4. RESULTS OF CALCULATION AND DISCUSSION

4.1 *Variation of the stress and the strain state*

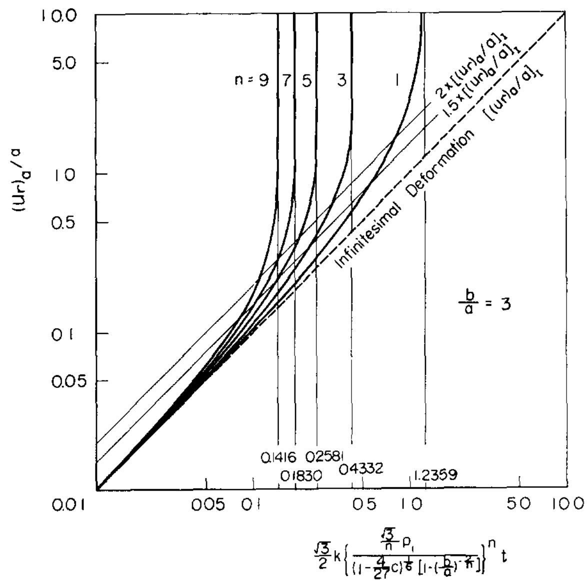
Now, let us examine the creep process of thick-walled tubes with closed ends under constant internal pressure p_i by using the preceding equations.

Figures 1 and 2, to begin with, show the variation of the radial displacement and the equivalent stress at inner radius for some values of b/a and n. Dashed lines in these figures represent the results of infinitesimal deformations, and vertical lines show the critical time t_∞ corresponding to $(u_r)_a \to \infty$ and $(\sigma_e)_a \to \infty$.

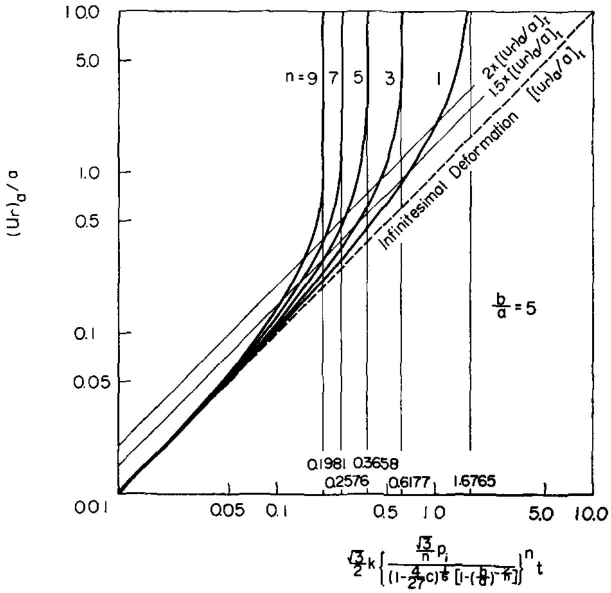
As observed in Fig. 1, the deviation of the results of the infinitesimal theory (thick dashed



1(a)



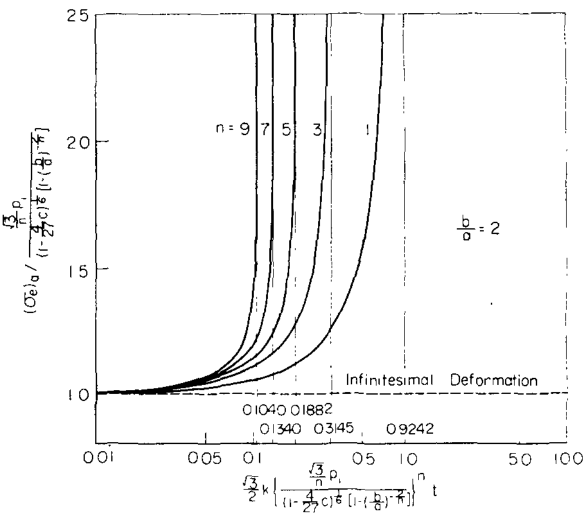
1(b)



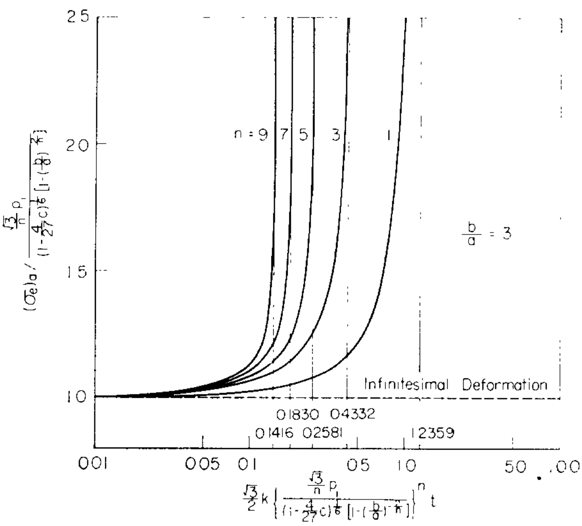
1(c)

Fig. 1. Radial displacement at the inner radius. (a) $b/a = 2$, (b) $b/a = 3$, (c) $b/a = 5$.

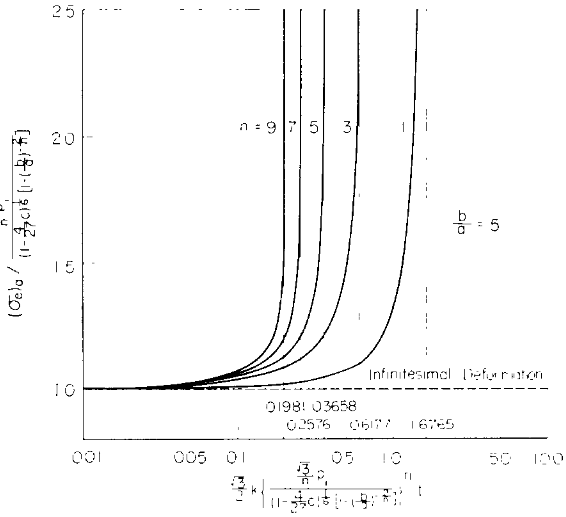
lines) from those of the present analysis (thick solid lines) increases rapidly as the critical time is approached, and its magnitude is more significant for larger values of creep exponents. Thin lines parallel to dashed lines represent values 1·5 and 2 times as large as the values of the dashed lines $[(u_r)_a]_I$. Thus, the intersections of these thin lines with the solid lines represent the state at which differences between thick solid lines and dashed lines are 50 and 100 per cent of the latter. Therefore, a given value of the deviation between the finite and the infinitesimal theory occurs at smaller displacement for the larger value of n. Thus, it will be observed that the physical non-linearity may increase effects of geometrical non-linearity on the creep of tubes. Similar tendencies are observed in Fig. 2 which indicates the variation of equivalent stress at the inner radius of the tube.



2(a)



2(b)



2(c)

Fig. 2. Equivalent stress at the inner radius. (a) $b/a = 2$, (b) $b/a = 3$, (c) $b/a = 5$.

The critical time t_∞ as shown by vertical lines in Figs. 1 and 2, corresponds to the rupture time t_B of an ideal process in which the cross section of the tube decreases continuously to zero and the tube remains a right cylinder until the last instant of the deformation. If the tube material is ductile and the applied pressure is not so small, the first of these conditions is satisfied practically[27], and does not cause any significant difference between t_∞ and t_B. However, in order to examine the feasibility of the second condition, it is necessary to elucidate the uniqueness and the stability bound of the relevant creep deformation[3], but such investigation has not been done for the non-linear steady-state creep deformation discussed here.

Finally, the relations between the critical time t_∞ and the parameters n and b/a are shown in Fig. 3. Since the critical time for the fixed values of b/a is the smaller the larger the value of n, the non-linearity in the creep law is again found to magnify the effect of geometrical non-linearity.

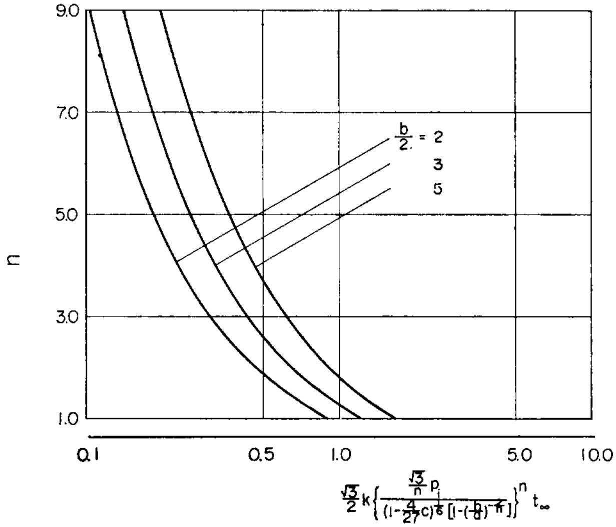


Fig. 3. Relation between the critical times, the creep powers and the tube geometries.

4.2 *Effects of the third invariant of deviatoric stress tensor*

Though the contribution of J_3 is much smaller than that of J_2, the effect of the former on the stress–creep rate relation is not always negligible in case of significant physical non-linearity. In fact, these authors have shown that the discrepancy between the analytical and experimental results for infinitesimal creep deformation may be partly attributed to the effect of J_3[23]. As observed from equations (47)–(50) or (54)–(58), the effect of J_3 for the case in question is reflected only through the time t. In case of $c \geqslant 0$, in particular, the effect of J_3 diminishes the value of t corresponding to a given magnitude of the deformation. The ratio between the critical times by the Prager–Drucker and Mises type theory (i.e. with and without regard to J_3) is $[1 - (4/27)c]^{n/6}$. The ratios are summarized in Table 1 for the case of $c = 2{\cdot}25$, which is the largest value consistent with the creep potential (20), and has been ascertained to correlate the experimental results for plastic deformations of aluminum alloy[31]. The effect of J_3 is significant for larger values of n, and in the case of $n = 9$ it reduces the critical time by a factor of about 2. Accordingly, disregard of the effects of J_3 may cause predictions on the unsafe side in the creep design of structural elements.

Table 1. Comparison between the critical times obtained from the Prager–Drucker and the Mises type theory

n	1	3	5	7	9
$(t_\infty)_{PD}/(t_\infty)_M$	0·9347	0·8165	0·7133	0·6231	0·5443

4.3 *Comparison with experimental results*

Results of the preceding analysis will herewith be compared with those from the experiment of Taira and Ohtani on tubes of 0·19 per cent carbon steel subject to internal pressure at 500 °C[32]. The inner and the outer diameter of the undeformed tubes were $2a = 25{\cdot}40$ and $2b = 50{\cdot}45$ mm. The results of tensile creep test on the tube material under nominal stress of $S = 10$, 12 and 14 kg/mm^2 are shown by the open circles in Fig. 4.† The solid lines in the figure show the approximate curves for the experimental results by equation (18) together with the material constants

$$k = 6{\cdot}38 \times 10^{-11}\ \text{hr}^{-1}(\text{kg/mm}^2)^{-6{\cdot}40}, \qquad n = 6{\cdot}40. \tag{62}$$

The curves were calculated by taking account of the decrease of the cross-sectional area due to the elongation and by assuming the constant volume of the specimens.

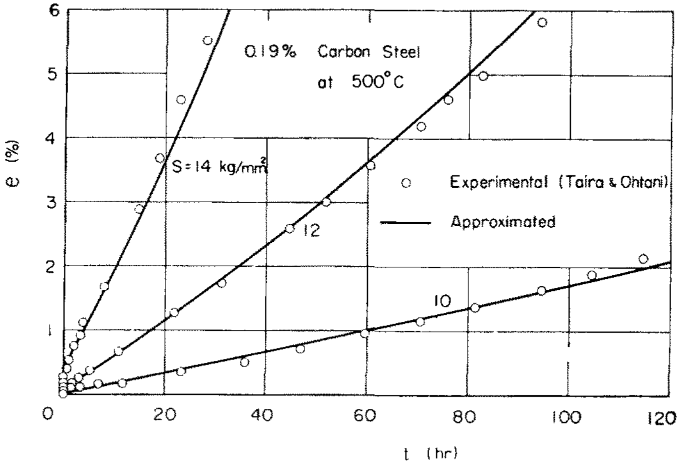


Fig. 4. Results of the calibration tests for the tube material.

The radial displacements at the outer radius in the tube tests are shown in Fig. 5, where the solid and the dash-and-dot lines show numerical results according to the Prager–Drucker ($c = 2{\cdot}25$) and the Mises ($c = 0$) type theory. The dashed lines in the figure represent the results of the analysis of infinitesimal deformations. As observed in the figure, the solid lines coincide with the experimental results with an accuracy of 20 per cent, while dash-and-dot lines show considerable deviation from them. Therefore, it may be concluded that the disregard of J_3 may cause pronounced underestimation of the deformation of structural elements and hence result in overestimation of their operating life.

The results by the infinitesimal deformation theory, on the other hand, deviate remarkably from the experimental results when the value of $(u_r)_b/b$ is in excess of about 0·01. For the larger value of p_i, in particular, these discrepancies are much larger than those caused by the difference between the creep laws. Thus, the geometrical non-linearity, in addition to physical non-linearity has an essential significance in creep analyses which are generally concerned with large deformations.

† Unpublished (according to a private communication).

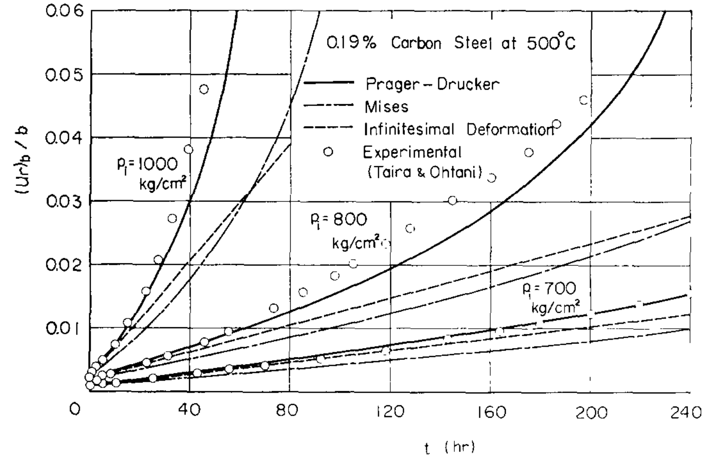


Fig. 5. Comparison between the analytical and the experimental results for the radial displacement at the outer radius.

4.4 *Comparison with Rimrott's analysis*

Rimrott[1] has analysed the relevant problem in the case of internal pressure by assuming the creep theory of Mises type and the Norton law for the true stress and logarithmic creep rate. Although he took account of the simultaneous change of the tube radius and the wall thickness, Rimrott simply modified the usual procedure for infinitesimal deformations and did not follow the general theory of finite deformations. Thus, it would be of interest to elucidate the relation of Rimrott's analysis to the present one which conforms to the rigorous theory of finite deformations.

The logarithmic strain and the true stress employed by Rimrott are related to the Cauchy stress tensor and the Cauchy–Green strain tensor in this paper by the relations

$$\left.\begin{aligned}&\varepsilon_{rr} = \tfrac{1}{2}\ln(2\gamma_{11}+1), \quad \varepsilon_{\theta\theta} = \tfrac{1}{2}\ln\left(2\frac{\gamma_{22}}{r^2}+1\right), \quad \dot{\varepsilon}_{rr} = \eta^{1}_{\cdot 1}, \quad \dot{\varepsilon}_{\theta\theta} = \eta^{2}_{\cdot 2},\\ &\sigma_{rr} = \tau^{1}_{\cdot 1}, \quad \sigma_{\theta\theta} = \tau^{2}_{\cdot 2}, \quad \sigma_{zz} = \tau^{3}_{\cdot 3}.\end{aligned}\right\} \tag{63}$$

Comparing equation (38) in Rimrott's paper

$$\int_0^{\varepsilon_a}\left\{1-\left[\frac{\left(\frac{a}{b}\right)^2 e^{\sqrt{3}\,\varepsilon_a}}{1+\left(\frac{a}{b}\right)^2(e^{\sqrt{3}\,\varepsilon_a}-1)}\right]^{1/n}\right\}^n d\varepsilon_a = \left(\frac{\sqrt{3}}{n}\right)^n k\int_0^t p_i^n\, dt$$

with equation (50) in this paper, we have the relation

$$\sqrt{3}\int_0^{\varepsilon_a}\left\{1-\left[\frac{\left(\frac{a}{b}\right)^2 e^{\sqrt{3}\,\varepsilon_a}}{1+\left(\frac{a}{b}\right)^2(e^{\sqrt{3}\,\varepsilon_a}-1)}\right]^{1/n}\right\}^n d\varepsilon_a = \int_0^H\left\{(1+H)^{-1/n}-\left[\left(\frac{b}{a}\right)^2+H\right]^{-1/n}\right\}^n dH$$

for the case of $c = 0$ and $p_0 = 0$, where ε_a denotes the equivalent strain at the inner surface of the tube. The above relation is satisfied identically by the relation

$$e^{\sqrt{3}\varepsilon_a} = 1 + H. \tag{64}$$

Therefore, the results of the present analysis for $c = 0$ and $p_0 = 0$ reduce to those of Rimrott under the condition of equation (64).

Actually, as he specified the equilibrium of the tube element at the deformed state and the true stress employed therein is identical with the physical components of Cauchy stress in this analysis, the equation of equilibrium and the definition of stress in both analyses are identical. The Norton law, expressed in terms of the true stress and the logarithmic strain rate used by Rimrott, furthermore, coincides again with that for Cauchy stress and the Cauchy–Green strain rate in this paper (see the footnote of p. 1204). Finally, with respect to the creep theory, equation (26) reduces to the Mises type theory when $c = 0$. Thus, all fundamental relations in both analyses are found to be identical. However, this holds true only in cases where a given problem can be solved with respect to the principal axes of stress and strain. Therefore, it should be noticed that an extension of the usual analytical procedures for infinitesimal deformation to the problem of finite deformations as performed by Rimrott is possible only for relatively simple and special problems like the present one.

5. CONCLUSION

The steady-state creep of pressurized thick-walled tubes was analysed by the theory of finite deformation. A state of plane strain was found to be realized rigorously also in the case of finite deformations as far as the creep potential is a function of J_2 and J_3 of even order with respect to stresses. The closed form solutions derived in the analysis elucidated that the physical non-linearity magnifies the effect of the geometrical non-linearity. The effect of J_3 was the more significant the larger the values of n, and in case of $n = 9$ it reduced the critical time by a factor of about 2. Results of the analysis agreed with those from the experiments with an accuracy of 20 per cent, while the results disregarding J_3 and those of infinitesimal deformation lead to considerable underestimation of the deformation. Rimrott's solution obtained by modifying the procedure for infinitesimal deformations practically coincided with a special case of the present solution.

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Абстракт — В качестве приложения теории конечных деформаций к задачам ползучести дается анализ стационарной ползучести толстостенных труб, находящихся под давлением. Выводятся в первый раз, конститутивные уравнения для случая конечных деформаций, путем предположения потенциала Прагера-Дракера и закона Нортона. Получаются решения в замкнутом виде. Сравниваются они, как с соответствующим экспериментом так и с анализом, который пренебрегает эффект инварианта J_3. Исследуется, также, зависимость между предлагаемым анализом и анализом Римротта, выполненным путем видоизменения процесса инфинитеземкльных деформаций.